SPIN VERSUS HELICITY

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We are express the fermion's probability amplitude as a trace over spinor indices but using the spin instead of the helicity thus retrieving in another way all results of the helicity formalism in simpler forms. Our spin formulation is however more appropriate to processes of transverse polarizations now of increasing popularity. The helicity formalism may do the work but only indirectly using redundant intermediary steps. On the other hand certain observables such as dipole moments electric or magnetic are naturally expressed in terms of spin rather than helicity The helicity formalism for its part, applies more to collision processes with specified helicities. But both formalisms have the advantage of making the amplitudes or even the squared amplitudes easily computable either analytically or symbolically, a consequence of the rewriting of the amplitudes as an overall trace over gamma matrices.

1. Motivating the spin

Amplitudes of processes involving fermions of spin one half are generally written as

$$\bar{u}(k,s)...X.....u(k',s') \quad (1)$$

where $u(k,s)$ is the part of the wave function which describes the spin of a particle of energy momentum p and spin s and where ellipses indicate other Dirac spinors. The probability for a given process to occur is the squared modulus of the amplitude. It is usually expressed as a trace over spinor indices by use of projectors

$$Tr(\bar{X}\frac{\not{p}'+m}{2m}\frac{1+\gamma_5\not{s}'}{2}\cdots X\cdots\frac{\not{p}+m}{2m}\frac{1+\gamma_5\not{s}}{2}) \quad (2)$$

The trace form of the probability is compact, Lorentz invariant and, in addition offers the possibility to handle several γ-matrices (several loops) using machine facilities. Several symbolic programs are made available for such symbolic computations. In the next section we will show that amplitudes similarly to probabilities are re-expressed as traces, and hence benefit from the same computational facilities as probabilities. This work has already been done within the helicity formalism [1]. Here we develop it within the spin formalism which is shown to be equivalent and also serves different purposes. One may ask the question why the spin, as the helicity did all. The helicity deals naturally with processes where the states are helicity states. States polarized in the direction (θ, ψ) are also dealt with but in an indirect way, as such states are linear combination of states of definite helicity

$$|\chi_s\rangle = \cos\frac{\theta}{2}|\chi_+\rangle + s\sin\frac{\theta}{2}e^{i\psi}|\chi_-\rangle \quad (3)$$

So treating general polarizations by the helicity formalism amounts to rewrite such polarizations in terms of helicity states according to (3) and this complicates the analysis. In the contrary the spin formalism treats such polarization directly as in (2). On the other hand, pure transverse polarizations (set θ = π/2 in (3)) are no longer unwanted polarizations since our launching of the transversity [2], in 1992 showing the relevance of the concept in hadronic physics. Away from the transversality in hadron physics, where the formalism of spin is more appropriate, we just want to calculate this: The scattering amplitude from the transversely polarized state

$$\begin{aligned}&|\chi_{s'}\otimes\bar{\chi}_s\rangle\\ &=\frac{1}{2}(|\chi_+\otimes\bar{\chi}_-\rangle\\ &+e^{i\bar{\psi}}s|\chi_+\otimes\bar{\chi}_+\rangle+e^{i\psi}s'|\chi_-\otimes\bar{\chi}_-\rangle\\ &+ss'e^{i(\psi+\bar{\psi})}|\chi_-\otimes\bar{\chi}_+\rangle)\end{aligned} \quad (4)$$

of the electron-positron system to a final state $|f\rangle$

$$\begin{aligned}&< f|T|\chi_{s'}\otimes\bar{\chi}_s >\\ &=\frac{1}{2}(T_{+-}+e^{i\bar{\psi}}sT_{++}+e^{i\psi}s'T_{--}+e^{i(\bar{\psi}+\psi)}ss'T_{-+})\end{aligned} \quad (5)$$

So if we use the helicity formalism for this process we should compute the expression above (5), which consists of four helicity amplitudes written as traces over gamma matrices of the form

$$T_{\lambda'\lambda} = Tr(\rho_{\lambda'\lambda}X) \quad (6)$$

But the use of the spin formalism amounts to compute only one term of the form (2) which is compact and more tractable

$$T_{s's} = Tr(\rho_{s\bar{s}}(k',k)\mathrm{X}) \quad (7)$$

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with $\rho_{s\bar{s}}(k',k)$ the generalized spin projector. It is the aim of the present study to give the full expression for $\rho_{s\bar{s}}(k',k)$ in order to deal with amplitudes of general polarizations (transverse polarization is a particular case) and to avoid computing various helicity amplitudes. A helicity approach to a process of general polarization is still more cumbersome if the final state is generally polarized. To cite just one example we take the scattering amplitude from the transversely polarized state of the electron-positron (or quark-antiquark) system to a final state of say two photons also polarized transversely. The helicity formalism necessitates computing the sixteen helicity amplitudes $T^{h\bar{h}}_{\lambda\bar{\lambda}}$ with $h,\bar{h}$ the helicity of the photons. Even if some symmetries are present such as chirality or parity (usually absent in supersymmetic models) this will not reduce the number of helicity amplitudes notably. There is another issue which necessitates a treatment within the spin framework as it depends directly on the transverse spin, namely the convection current part of the quark dipole magnetic moment which can ultimately be written as (see section 3)

$$\propto \int |\vec{k}|^2 \, \vec{s}_\perp(k) \frac{d^3k}{(2\pi)^3} \tag{8}$$

These considerations show definitely that the prejudice against the spin formalism as being an unessential formalism as compared to the helicity formalism has no raison d'être and this is sufficient to motivate the present study.

2. The probability amplitude as a trace

Write the spinor amplitude as

$$Tr(\cdots f\rho)$$

$$\rho_{\alpha'\alpha}(k',s',k,s) = u_{\alpha'}(k',s')\bar{u}_\alpha(k,s) \tag{9}$$

The approach we follow to work out the expression for the projector ρ is to extract it from its primitive form at the rest frame (where it is relatively easy to compute) by performing a Lorentz boost. The form of the generalized density is (from now on we hide Dirac indices)

$$\begin{aligned}&\rho(k',s',k,s)\\&= \frac{2\mathrm{N}_{s's}}{1+ss'\vec{\zeta}.\vec{\zeta}'}\left(\frac{\not{k}'+m'}{2m'}\right)\left(\frac{1+s'\gamma^5\not{s}'}{2}\right)\\&\Re(\vec{k}',\vec{k})\left(\frac{\not{k}+m}{2m}\right)\left(\frac{1+s\gamma^5\not{s}}{2}\right)\end{aligned} \tag{10}$$

The operator $\Re(\vec{k}',\vec{k})$ flips the momentum from $\vec{k}$ to $\vec{k}'$ while $\mathrm{N}_{s's}$ a matrix in the two dimensional space of solutions ($s=\pm 1$ is twice the spin) is responsible for the spin flip making the passage from the state $u_\alpha(k,s)$ to the state $u_\alpha(k',s')$ computed in the rest frame The matrix $\Re$ will be shown to have the explicit form

$$\begin{aligned}&\Re(\vec{k}',\vec{k})\\&= \exp\left(-\frac{\omega'}{2}\gamma_0\frac{\vec{\gamma}.\vec{k}'}{|\vec{k}'|}\right)\exp\left(\frac{\omega}{2}\gamma_0\frac{\vec{\gamma}.\vec{k}}{|\vec{k}|}\right)\\&= \cosh\left(\frac{\omega'}{2}\right)\cosh\left(\frac{\omega}{2}\right)\\&-\sinh\left(\frac{\omega'}{2}\right)\cosh\left(\frac{\omega}{2}\right)\gamma_0\frac{\vec{\gamma}.\vec{k}'}{|\vec{k}'|}\\&+\sinh\left(\frac{\omega}{2}\right)\cosh\left(\frac{\omega'}{2}\right)\gamma_0\frac{\vec{\gamma}.\vec{k}}{|\vec{k}|}\\&+\sin h\left(\frac{\omega'}{2}\right)\sin h\left(\frac{\omega}{2}\right)\frac{\vec{\gamma}.\vec{k}'}{|\vec{k}'|}\frac{\vec{\gamma}.\vec{k}}{|\vec{k}|}\end{aligned} \tag{11}$$

with $\omega = -\tanh^{-1}\left(\frac{|\vec{k}|}{k^0}\right)$ and idem for ω'. Further simplifications lead to the formula

$$\Re = \sqrt{\frac{mm'}{(k_0+m)(k'_0+m')}}(\gamma_0+1) \tag{12}$$

3. Computing a given process within the formalism

Let us compute the process $e^+e^- \to \tilde{e}^+_R\tilde{e}^-_R$ of the creation of right-handed selectrons by electrons via exchange of photinos in the chiral case (photons, Z bosons and zino exchanges are not considered as this computation is just an illustration).The positron has momentum k along the z axis in the e^+e^- centre of mass and spin projection $-s$ along the transverse direction η_1 while the electron has momentum k' and spin $-s$ along η'_1 opposite to η_1 (This is the natural polarization of the system in in the storage ring.) The momentum of the selectron is $p=(p^0,\vec{p})$ with $\vec{p}=|\vec{p}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. We may replace the electron-positron system by the quark-antiquark system with transverse polarization inherited from the parent proton-antiproton system transversely polarized.

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For the electron-photino vertex, we use that fixed by super-symmetry in the chiral case

$$\sqrt{2}\, e\bar{\tilde{\gamma}}\frac{(1+\gamma_5)}{2}e\tilde{e}*_R ,$$

but before to compute the amplitude we have to include the anti- particle into the formalism since so far we have considered only the projector involving particles. To do so we adopt the convention that the anti-particle spinor is related to the particle spinor by the relation [3]

$$v(k,\lambda) = -\lambda\gamma_5 u(k,-\lambda) \tag{13}$$

Applying the relation (13) to the spinor $u(k,s)\,|_{\vec{\zeta}=\vec{\eta}_1}$ we get

$$\begin{aligned} u(k,s)\,|_{\vec{\zeta}=\vec{\eta}_1} &= \frac{u(k,1)+su(k,-1)}{\sqrt{2}} \\ &== s\gamma_5\frac{v(k,1)-sv(k,-1)}{\sqrt{2}} \\ &= s\gamma_5 v(k,-s) \end{aligned} \tag{14}$$

In what follows all quantities will be computed using the energy projectors $\not{k}+m$ (suitable for the massless limit or high energy which we will adopt here).

The amplitude computed at vanishing azimuthal angle is of the form

$$\begin{aligned} &2e^2\bar{v}(k,-s)\frac{(1-\gamma_5)}{2}(\frac{\not{k}'-\not{p}+m_{\tilde{\gamma}}}{t-m^2_{\tilde{\gamma}}})\frac{(1+\gamma_5)}{2}u(k',-s) \\ &= -2se^2\frac{(1-\gamma_5)}{2}(\frac{\not{k}'-\not{p}+m_{\tilde{\gamma}}}{t-m^2_{\tilde{\gamma}}})\frac{(1+\gamma_5)}{2}u(k',-s)\bar{u}(k,s) \\ &= \frac{se^2}{2(t-m^2_{\tilde{\gamma}})}Tr(1-\gamma_5)(\not{k}'-\not{p})\gamma_0\not{k}(1+s\gamma^5\not{\eta}_1) \end{aligned} \tag{15}$$

with $t=(k'-p)^2$. The γ_5 term is vanishing as it is proportional to $\vec{p}.(\vec{k}'\times\vec{\eta}_1)=\vec{p}.\vec{\eta}_2=0$ while the remaining term is

$$\begin{aligned} &\frac{e^2}{2(t-m^2_{\tilde{\gamma}})}Tr(\not{k}'-\not{p})\gamma_0\not{k}\not{\eta}_1 \\ &= \frac{2e^2k_0^2\beta_R\sin\theta}{(t-m^2_{\tilde{\gamma}})} \end{aligned} \tag{16}$$

with $\beta_R=\frac{|\vec{p}|}{p^0}$ the velocity of the selectron. This is a one-step computation in the spin formalism. To compute the same amplitude but in the helicity formalism as is commonly practiced so far, we have to compute four helicity amplitudes separately. The amplitude of the above process computed in the helicity formalism involves the helicity quantities

$$\begin{aligned} A_\mu &= \bar{u}(k',\lambda')\gamma_\mu v(k,\lambda) \\ B_\mu &= \bar{u}(k',\lambda')\gamma_\mu\gamma_5 v(k,\lambda) \end{aligned} \tag{17}$$

Computed in our convention, they lead to the expressions

$$\begin{aligned} A_{\mu,\lambda,\lambda'} &= \begin{cases} -i2m\lambda' g_{\mu i}\hat{k}'^{i} & \lambda=\lambda' \\ -i2k'_0\,\eta^\lambda_\mu & \lambda=-\lambda' \end{cases} \\ B_{\mu,\lambda,\lambda'} &= \begin{cases} -i2\lambda'\left|\vec{k}'\right|\eta^\lambda{}_\mu & \lambda=-\lambda' \\ -i2mg_{\mu 0} & \lambda=\lambda' \end{cases} \end{aligned} \tag{18}$$

The helicity amplitudes turn out to all vanish in the limit of vanishing electron mass, except the one associated to the right-handed electron which we write explicitly

$$\begin{aligned} &2e^2\bar{v}(k,-1)\frac{(1-\gamma_5)}{2}(\frac{\not{k}'-\not{p}+m_{\tilde{\gamma}}}{t-m^2_{\tilde{\gamma}}})\frac{(1+\gamma_5)}{2}u(k',1) \\ &= \frac{e^2(k'-p)^\mu}{t-m^2_{\tilde{\gamma}}}(A^*{}_{\mu,1,-1}+B^*{}_{\mu,1,-1}) \\ &= -ie^2\frac{4k_0\,p.n^\lambda_\mu}{t-m^2_{\tilde{\gamma}}} \\ &= -ie^2\beta_R\frac{4k^2{}_0\sin\theta}{t-m^2_{\tilde{\gamma}}} \end{aligned} \tag{19}$$

The transversely polarized amplitude (16) is then recovered using the decomposition of the spin projector $\tilde{\rho}_{\alpha,\alpha}{}'=u(k,\alpha)\bar{v}(k',\alpha')$ with $\alpha=s,\lambda$

$$\tilde{\rho}_{-s,-s} = -\frac{i}{2}\left[s(\tilde{\rho}_{1,1}-\tilde{\rho}_{-1,-1})+\tilde{\rho}_{-1,1}-\tilde{\rho}_{1,-1})\right] \tag{20}$$

The transversely polarized amplitude is then according to (20)

$$\begin{aligned} \frac{i}{2}Tr(\tilde{\rho}_{1,-1}\cdots) &= \frac{i}{2}(-ie^2\beta_R\frac{4k^2{}_0\sin\theta}{t-m^2_{\tilde{\gamma}}}) \\ &= \frac{2e^2k_0^2\beta_R\sin\theta}{t-m^2_{\tilde{\gamma}}} \end{aligned} \tag{21}$$

4. The quark dipole magnetic moment

By using the Gordon decomposition we divide the dipole magnetic moment expression into two terms: one is the convection current part and the other is the spin

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part. As an application of our formalism we compute the convection part as it is the only part which involves the generalized spin density. The convection current part has the form

$$-i\int\left[Tr\vec{\nabla}_k\rho\right]_{\substack{\vec{k}'=\vec{k}\\ \vec{\zeta}'=\vec{\zeta}}}\times\vec{k}\,\frac{d^3k}{(2\pi)^3}$$

$$=\frac{i}{2(k_0+m)}$$

$$\int(\vec{k}\times Tr\left[(\frac{\not{k}+m}{2m})(\frac{1+\gamma_5\not{s}}{2})(\gamma_0\vec{\gamma})\right])\frac{d^3k}{(2\pi)^3}$$

$$=\frac{-i}{8m(k_0+m)}\int(\vec{k}\times Tr\gamma_5\not{k}\not{s}\gamma_0\vec{\gamma})\frac{d^3k}{(2\pi)^3}$$

$$=-\frac{1}{2m(k_0+m)}\int|\vec{k}|^2\,\vec{s}_\perp(k)\frac{d^3k}{(2\pi)^3}$$

(22)

Note the natural occurrence of the transverse spin $\vec{s}_\perp$ in the calculation of the convection current. If such current was computed in the helicity formalism we would have get a more complicated expression. This has to be compared with the lengthy computation of the same observable using the Dirac spinors [4].

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